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THE LIMITING PROPERTIES OF PIECEWISE-POTENTIAL SUBCRITICAL AND CRITICAL JET STREAMS OF AN IDEAL GAS†

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The limiting properties of subcritical and critical (with Mach numbers $M \leqslant 1$) plane-parallel jet streams are investigated in the approximation of an ideal (inviscid and non-heat-conducting) gas. Chaplygin's equation is used with the pressure and the angle of inclination of the velocity as the independent variables, which are measured from the limiting values corresponding to the cross-section of the equalizing of the jet with respect to these variables. The stream function in the neighbourhood of the "equalizing cross-section" is represented in the form of an expansion in powers of the "distance to the origin of the coordinates" (in the plane of the independent variables) with coefficients which depend on a previously unknown combination of the independent variables. The limiting property of the flow, i.e. the position of the equalizing cross-section at a finite of infinite distance, is defined by the leading terms of the expansion. A symmetric potential jet and symmetric piecewise-potential jets flowing out into a submerged space in the case of subcritical and critical pressure drops are considered as examples. The critical pressures of the potential parts of the composite jet can be different or identical (including when the thermodynamic properties of the gases are different).

1. INTRODUCTION

Following Chaplygin [1], a change to the value V and the angle of inclination θ of the velocity vector $\mathbf{V}$ as the independent variables, and the stream function ψ, the potential and the Cartesian coordinates x and y as the dependent variables is widely used in problems of plane-parallel potential jet streams of an ideal gas. In a number of problems with impermeable boundaries consisting of rectilinear segments, the resulting linear equation for ψ, with coefficients which depend solely on V ("Chaplygin's equation"), admits of separation of the variables and a solution as a whole in the form of infinite series [1–10]. However, in such cases, quite often a complete solution of a problem cannot be successfully obtained since, in the cases of gases with real thermodynamics, the equations for the infinite set of functions which depend on V can only be integrated numerically. Even in the case of a perfect gas with constant heat capacities, for which these functions turn out to be well-known solutions of the hypergeometric equation, the summation of the corresponding series is an exceedingly complex problem (see [8] and [9, pp. 228–234]).

It is simpler to investigate the limiting properties of the above-mentioned flows by which we mean the position of the equalizing cross-section of the jet parameters because, in order to do this, calculation of the sums of the corresponding series is not required, but only an investigation of their convergence or divergence. In fact, it has been proved in [1] that, for a subcritical pressure drop (the pressure in the submerged space is higher than the critical pressure of the jet) in the case of a perfect gas, the equalizing is of an asymptotic character, as in the case of an incompressible fluid. It has been shown [2] (see also [4]) that subcritical jets of an arbitrary barotropic gas possess the same property.

Whereas the asymptotic character of the equalizing of subcritical jets is natural, the equalizing of a critical jet in a straight sonic line at a finite distance from the section of the nozzle, which was established for the first time in [3], turned out to be unexpected. This property was obtained [3] for a perfect gas in a problem which permitted separation of the variables as a consequence of the convergence of the corresponding series. An extension of the result in [3] to an arbitrary barotropic gas, carried out in [2], was presented in [4]. Using the same method, the result in [3] was extended [5, 8–10] to the case of a sonic jet of a perfect gas flowing over wedge-shaped obstacles. In this flow problem, two straight sonic lines are formed bounding the finite domain of subsonic flow from above and from below along the stream. Note that the rectilinearity of the sonic lines in two-dimensional (plane-parallel and axial-symmetric) potential flows with $M \leqslant 1$ also follows from a theorem proved in [11] (also, see [12]).

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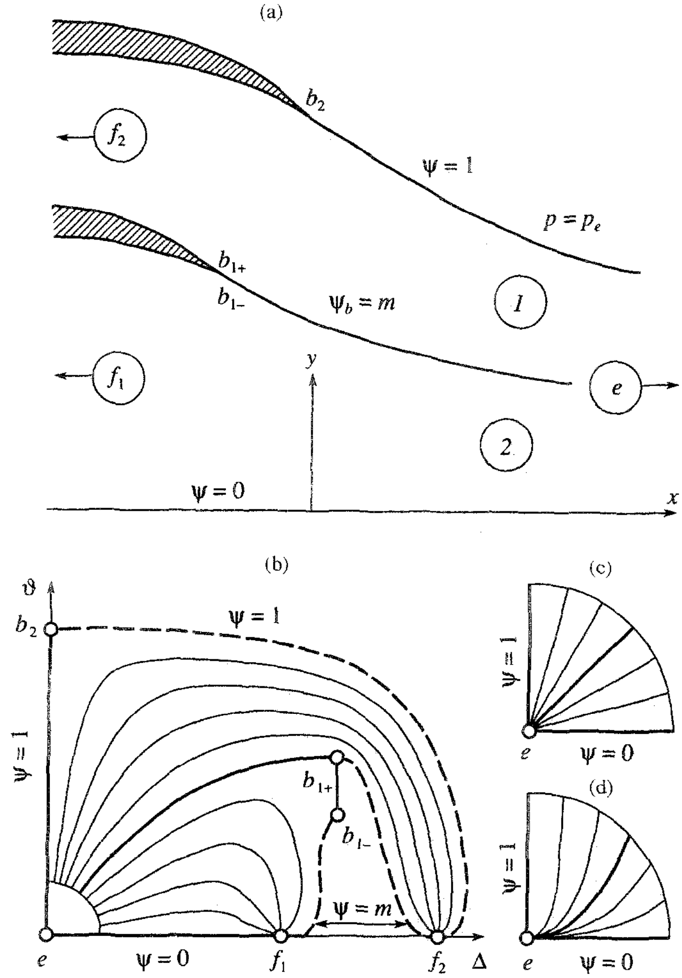

Fig. 1

Below, in a development of the results which have previously been obtained [13], a method for investigating the limiting (equalizing) properties of plane-parallel potential and piecewise-potential jets is described which is simpler than the analysis of the convergence of the series. The velocity is disrupted at the tangential discontinuities, which separate the piecewise-potential jets. Chaplygin's equation is therefore used in a form in which the modulus of the velocity V is replaced by the pressure p. The method rests on an analysis of the structure of the solution in the neighbourhood of the singular point in the plane of the independent variables, p and angle $\vartheta = -\theta$, which corresponds to the equalizing cross-section. This analysis leads to an ordinary differential equation which is integrated in quadratures. It is not necessary when using this method that the problem as a whole should allow of separation of the variables nor that the gas should be a perfect gas.

2. STRUCTURE OF THE SOLUTION IN NEIGHBOURHOOD OF THE SINGULAR POINT. HOMOGENEOUS POTENTIAL JET

Consider a plane-parallel jet of an ideal gas which flows out of a nozzle (Fig. 1a) into a submerged space with a pressure p_e. The form of the jet is quite arbitrary, apart from the existence of a plane of symmetry and the absence of segments of generatrices, around which a flow could give rise to local supersonic zones with discontinuities which upset the isentropic character of the flow. As in Fig. 1(a), the nozzle can have internal walls. In the case corresponding to Fig. 1(a), two jets of, generally speaking, different gases with distinct stagnation enthalpies H and specific entropies S flow over the plane of symmetry along which we direct the x axis of the Cartesian coordinates x and y. When H and S for each jet are homogeneous, the flow in them is a potential flow. This enables us to change to the variables V

and θ, where θ is the angle of inclination of the velocity vector to the x axis. In these variables, for an arbitrary two-parameter gas, the stream function satisfies Chaplygin's equation [7]

$$V^2\psi_{VV} + V(1+M^2)\psi_V + (1-M^2)\psi_{\theta\theta} = 0 \tag{2.1}$$

and the equality

$$dx = \left(\frac{M^2-1}{\rho V^2}\psi_\theta\cos\theta - \psi_V\sin\theta\right)dV + \frac{1}{\rho V}(V\psi_V\cos\theta - \psi_\theta\sin\theta)d\theta \tag{2.2}$$

holds for the differential dx.

In Eqs (2.1) and (2.2), all the variables are dimensionless. In the reduction to dimensionless form, we took p_e°, V_{*k}° and $p_e^\circ/(V_{*k}^\circ)^2$ as the scales of pressure, velocity and density. Dimensionless quantities are labelled with a "degree symbol", critical parameters (the critical velocity here) are labelled with an asterisk subscript and the jet parameters, which are numbered from the plane of symmetry (K is the number of piecewise potential jets), are labelled with the subscripts $k = 1, \dots, K$. Since the pressure scale, the pressure in the submerged space p_e°, is the same for the whole flow, the velocity and density scales for each jet have no effect on the fields of p and $\vartheta = -\theta$ which are subsequently important and continuous at the tangential discontinuities. The additive constant and the normalization factor which are permissible when introducing the stream function are chosen such that, for $\psi = \psi(p, \vartheta)$,

$$\psi(p, 0) = 0, \quad \psi(1, \vartheta) = 1 \tag{2.3}$$

which has taken account for the fact that $p = p_e = 1$ on the boundary of the jet with the submerged space.

The equations and conditions (2.1)–(2.3) not only hold for a flow which is piecewise-homogeneous with respect to H° and S° and, as a consequence of this, potential flows but, also, for those substantially inhomogeneous flows which reduce to homogeneous or to piecewise-homogeneous flows. In the problems being considered, according to well-known results [14] (also, see [15]), this is a jet of a perfect gas with a stagnation enthalpy H° which depends on ψ and, consequently, $V_*^\circ = V_*^\circ(\psi)$ also in the case of a constant or piecewise-constant critical pressure p_*°. In the case of such flows, when the pressure, velocity and density scales are chosen in a similar way (with its own V_*° in each streamline), the equations of motion for the dimensionless (without "degree symbols") and dimensional parameters, written for the independent variables x and y, only differ in the presence of the "degree symbols" as in the case of homogeneous flow. The condition for the flow to be potential flow is obtained from them as a consequence of the constancy of the now dimensionless H and S. In the case of a perfect gas with an adiabatic exponent κ, after changing, as described above, to dimensionless quantities, $H = (\kappa + 1)/[2(\kappa - 1)]$, and the constancy of S is equivalent to the constancy of the relation

$$p/\rho^\kappa = p_*/\rho_*^\kappa = = \kappa^{-\kappa}(p_*^\circ/p_e^\circ)^{1-\kappa} = \kappa^{-\kappa}p_*^{1-\kappa}$$

In connection with problems of the efflux of piecewise-homogeneous jets, we change from V to p, taking account of the fact that, in each homogeneous jet, p is solely a function of V, $dp/dV = -\rho V$ and $d^2p/dV^2 = \rho(M^2 - 1)$. Furthermore, we replace $\theta \leqslant 0$ by $\vartheta = -\theta \geqslant 0$ in Eqs (2.1) and (2.2). After this, we have

$$\psi_{pp} - \alpha\psi_p + \beta^2\psi_{\vartheta\vartheta} = 0, \quad \alpha = \frac{2}{\rho V^2}, \quad \beta^2 = \frac{1-M^2}{\rho^2 V^4} \tag{2.4}$$

$$dx = \left(\frac{\sin\vartheta}{\rho V}\psi_p - \beta^2 V\psi_\vartheta\cos\vartheta\right)dp + \left(V\psi_p\cos\vartheta + \frac{\sin\vartheta}{\rho V}\psi_\vartheta\right)d\vartheta \tag{2.5}$$

A certain domain of the plane Δ, ϑ with $\Delta = p - p_e = p - 1$ corresponds to the upper half of the potential and piecewise-potential jets escaping from the convergent nozzles of the type shown in Fig. 1(a). In the general case, when the whole of the boundary of this domain is unknown, it is subsequently only important (Fig. 1b) that the abscissa ($\vartheta = 0$) should correspond to the jet axis and that the ordinate ($\Delta = 0$) should correspond to the boundary with the submerged space. According to conditions (2.3), $\psi = 0$ on the abscissa and $\psi = 1$ on the ordinate. Hence, the origin of the coordinates $\Delta = \vartheta = 0$, which corresponds to the cross-section of the equalizing of the jet with respect to pressure and the direction of the velocity in the x, y plane, is the singular point at which all the streamlines with $0 \leqslant \psi \leqslant 1$ arrive. In Fig. 1(b), the points b, e, ... correspond to the same points of the x, y plane in Fig. 1(a), the unknown segments of the boundary of the flow are given by the dashes, the flow boundaries and the solid curves are the streamlines in which ψ = const. and the heavy line is the streamline separating the homogeneous jets. The nature of the equalizing is determined by the behaviour of the streamlines in the neighbourhood of the origin of the coordinates. The streamlines in this neighbourhood are not shown in Fig. 1(b). Two versions of the approach of the streamlines to the origin of the coordinates are depicted in Fig. 1(c, d).

Suppose $\delta = \delta(\Delta, \vartheta)$ is the distance to the singular point, which has been determined by some method or other. Then, in the neighbourhood of this point, it is natural to seek $\psi(\Delta, \vartheta)$ in the form

$$\psi = \psi_0(\chi) + \psi_1(\chi)\delta + \psi_2(\chi)\delta^2 + \dots, \quad \chi = \vartheta\Delta^{-n} \tag{2.6}$$

with an unknown exponent n. In using this expansion, it is necessary to distinguish between the cases of the escape of subcritical jets ($p_* < M_e < 1$ and $\beta_e > 0$) and critical jets ($p_* = 1$, $M_e = 1$ and $\beta_e = 0$).

On substituting expansion (2.6) into Chaplygin's equation in the form of (2.4) and taking account of the smallness of δ and Δ, we arrive at the equation

$$(n^2\chi^2 + \beta_e^2\Delta^{2(1-n)})\psi_0'' + n(n+1)\chi\psi_0' = 0 \tag{2.7}$$

for the "subcritical" jets in the leading orders with respect to δ and Δ.

We now consider different cases. At the singular point $\Delta = 0$ and, close to it, $\Delta \ll 1$. By virtue of this, certain terms in Eq. (2.7) can be neglected depending on the value of n. In accordance with this, when $M_e < 1$, the solutions (C_1 and C_2 are integration constants)

$$\begin{gathered} \psi_0 = C_1\chi^{-1/n} + C_2, \quad n < 1; \quad \psi_0 = C_1\chi + C_2, \quad n > 1 \\ \psi_0 = C_1 \operatorname{arctg}\frac{\chi}{\beta_e} + C_2, \quad n = 1 \end{gathered} \tag{2.8}$$

are possible for Eq. (2.7).

In the case of a homogeneous jet, χ varies from zero (on the jet axis, that is, on the Δ axis of the Δ, ϑ plane) to infinity (on the jet boundary, that is, in the ϑ axis of the same plane). The solution (2.8) with $n = 1$ therefore corresponds to a finite change in ψ_0 and ψ, and, taking account of conditions (2.3),

$$\psi_0 = \frac{2}{\pi}\operatorname{arctg}\frac{\chi}{\beta_e}, \quad \chi = \frac{\vartheta}{\Delta} \tag{2.9}$$

Figure 1(c) corresponds to this solution.

In the case of critical jets for which $p_* = 1$, $M_e = 1$ and $\beta_e = 0$, expanding $\beta = \beta(\Delta)$ with respect to Δ, we write Eqs (2.4), with an accuracy which is sufficient in what follows, in the form

$$\psi_{pp} - \alpha_*\psi_p + \left(\frac{3q}{2}\right)^2\Delta\psi_{\vartheta\vartheta} = 0, \quad \alpha_* = \frac{2}{\rho_*} = \frac{2}{\kappa}, \quad q^2 = \frac{-8}{9\rho_*^2}\left(\frac{dM}{dp}\right)_* = \frac{4(\kappa+1)}{9\kappa^3} \tag{2.10}$$

Here, the factor of $^3/_2$ in the brackets is introduced in order to simplify the subsequent formulae, and the second expressions for α_* and q^2 correspond to a perfect gas. Similarly, the expression for dx changes in the case of critical jets. Now, instead of expression (2.5), we shall have

$$dx = \left[\frac{\sin\vartheta}{\rho_*}\psi_p - \left(\frac{3q}{2}\right)^2\Delta\psi_\vartheta\cos\vartheta\right]dp + \left(\psi_p\cos\vartheta + \frac{\sin\vartheta}{\rho_*}\psi_\vartheta\right)d\vartheta \tag{2.11}$$

Substitution of expansion (2.6) into Eq. (2.10) leads to the equation

$$\left(n^2\chi^2 + \frac{9}{4}q^2\Delta^{3-2n}\right)\psi_0'' + n(n+1)\chi\psi_0' = 0 \tag{2.12}$$

Hence, as earlier, we find that, when $p_* = 1$ and $M_e = 1$

$$\psi_0 = C_1\chi^{-1/n} + C_2, \quad n < \frac{3}{2}; \quad \psi_0 = C_1\chi + C_2, \quad n > \frac{3}{2}$$
$$\psi_0 = C_1 I(\chi_0, \chi, q) + C_2, \quad I(a, b, c) = \int_a^b \frac{d\xi}{(\xi^2 + c^2)^{5/6}}, \quad n = \frac{3}{2} \tag{2.13}$$

In the case of a homogeneous jet, a finite change in ψ_0 and ψ gives a solution (2.13) with $n = 3/2$ and, when account is taken of conditions (2.3)

$$\psi_0 = \frac{I(0, \chi, q)}{I(0, \infty, q)}, \quad \chi = \frac{\vartheta}{\Delta^{3/2}} \tag{2.14}$$

Figure 1(d) corresponds to this solution.

In order to find the abscissa x_e of the cross-section of the equalizing of a potential (homogeneous) jet, we substitute expansions (2.6) and solutions (2.9) or (2.14) into expression (2.2), we retain the leading terms with respect to Δ and ϑ and integrate the resulting equation from a certain, "initial" cross-section, which is sufficiently distant from the nozzle section with respect to Δ from Δ_0 to $\Delta \to 0$ when $\vartheta = 0$ or, with respect to ϑ, from ϑ_0 to $\vartheta \to 0$ when $\Delta = 0$. Parameters in the chosen initial cross-section are labelled with a zero subscript, and integration when $\vartheta = 0$ gives x_e in the plane of symmetry of the jet and, when $\Delta = 0$, on its boundary. In order to determine the limiting ordinate of the boundary, y_e, that is, the half width of the equalized jet simultaneously with the calculation of x_e, we integrate the equation of a streamline

$$dy = (\operatorname{tg}\vartheta)dx = \vartheta(dx/d\vartheta)d\vartheta$$

with respect to ϑ. At the same time, $dx/d\vartheta$ is a calculated from Eq. (2.11) with $dp = 0$ and with solutions (2.9) or (2.14) for the stream function. Finally, we find that, in the case of a subcritical jet

$$x_e = x_0 - \frac{2}{\pi}V_e\beta_e \lim_{\Delta \to 0} \ln\frac{\Delta}{\Delta_0} = x_0 - \frac{2}{\pi}V_e\beta_e \lim_{\vartheta \to 0} \ln\frac{\vartheta}{\vartheta_0} \to \infty, \quad y_e = y_0 - \frac{2}{\pi}V_e\beta_e\vartheta_0^2 \tag{2.15}$$

and, for a critical jet

$$x_e = x_0 + \frac{9}{2I(0, \infty, q)}\left(\frac{3q}{2}\right)^{1/3}\Delta_0^{1/2} = x_0 + \frac{9}{2I(0, \infty, q)}\vartheta_0^{1/3} < \infty$$
$$y_e = y_0 - \frac{9}{8I(0, \infty, q)}\vartheta_0^{4/3} \tag{2.16}$$

The known properties of potential, plane-parallel jets of an ideal gas, which have already been mentioned, follow from formulae (2.15) and (2.16): asymptotic equalizing in the case of subcritical pressure drops ($p_* < 1$) and equalizing at a finite distance in the case of a critical drop ($p_* = 1$).

The first formulae for the coordinate x_e, (2.15) and (2.16), give it in terms of the pressure on the jet axis and the second formulae give it in terms of the angle of inclination of the boundary. According to formula (2.15) and the formula for dy preceding it, the relations

$$\frac{\Delta}{\Delta_0} = \frac{\vartheta}{\vartheta_0}, \quad x = x_0 - \frac{2}{\pi}V_e\beta_e\ln\frac{\Delta}{\Delta_0} = x_0 - \frac{2}{\pi}V_e\beta_e\ln\frac{\vartheta}{\vartheta_0}, \quad y = y_0 - \frac{2}{\pi}V_e\beta_e(\vartheta_0^2 - \vartheta^2) \tag{2.17}$$

are satisfied in the case of subcritical jets on the infinitely long equalizing segment (when Δ_0, Δ, ϑ_0 and $\vartheta \ll 1$).

Similarly, in the case of critical jets

$$\sqrt{\Delta_0} - \sqrt{\Delta} = \left(\frac{\vartheta_0}{q}\right)^{1/3} - \left(\frac{\vartheta}{q}\right)^{1/3}, \quad x = x_0 + \frac{9}{2I(0, \infty, q)}\left(\frac{3q}{2}\right)^{1/3}(\Delta_0^{1/2} - \Delta^{1/2}) =$$
$$= x_0 + \frac{9}{2I(0, \infty, q)}(\vartheta_0^{1/3} - \vartheta^{1/3}), \quad y = y_0 - \frac{9}{8I(0, \infty, q)}(\vartheta_0^{4/3} - \vartheta^{4/3}) \tag{2.18}$$

Formulae (2.17) and (2.18) in the equalizing segments ($\Delta < \Delta_0 \ll 1$ and $\vartheta < \vartheta_0 \ll 1$) of infinite or finite extension associate the coordinates x, y, x_0 and y_0 of the jet boundary with the angles of its inclination ϑ and ϑ_0, the abscissae x and x_0 of the jet axis with Δ and Δ_0 in it and, finally, ϑ and ϑ_0 in the boundary with Δ and Δ_0 on the axis. The relations indicated are universal as they do not depend on the form of symmetry of the nozzle. In the case of subcritical jets, the relation between ϑ and ϑ_0 on the boundary and Δ and Δ_0 on the axis is also independent of the gas properties.

If an inlet channel has a cylindrical part, then, for large negative values of x, solution (2.8) with $n = 1$ and with $\Delta = p - p_f$ also describes the flow in it. Here, the abscissa of the equalizing cross-section $x_f \to -\infty$.

Solutions (2.8) and (2.13) were also obtained in the case when, in expansion (2.6), one takes $\chi = \vartheta/f(\Delta)$ instead of $\chi = \vartheta/\Delta$ with a function $f(\Delta)$ which is unknown in advance, satisfying the condition $f(0) = 0$. By virtue of this condition, as before $\chi = \infty$ on the boundary of the jet and $\chi = 0$ in its plane of symmetry. In this case, for subcritical jets, Eq. (2.7), which determines $\psi_0(\chi)$, is replaced by

$$(\chi^2 f'^2 + \beta_e^2)\psi_0'' + (2f'^2 - ff'' + \alpha_e ff')\chi\psi_0' = 0 \tag{2.19}$$

Putting

$$ff'' = kf'^2 \tag{2.20}$$

we will investigate how the choice of the constant k affects the solution of Eq. (2.19). On integrating Eq. (2.20), we find that its solution satisfies the condition $f(0) = 0$ only for $-\infty < k < 1$. At the same time,

$$f = \Delta^n, \quad n = \frac{1}{1-k}, \quad f' = n\Delta^{n-1}, \quad f'^2 = n^2\Delta^{2(n-1)}, \quad f'' = n(n-1)\Delta^{n-2}$$

Hence, for different permissible values of k and, consequently, also n, "power" $\chi = \vartheta/\Delta^n$ and the previous solutions (2.8) and (2.13) are obtained.

3. EQUALIZING PROPERTIES OF PIECEWISE-POTENTIAL JETS

When considering two jets (Fig. 1a) with differing critical pressures and gas properties (the adiabatic exponents κ in the case of perfect gases) their parameters, according to what has been said above, are labelled with the subscripts 1 and 2. Each jet is homogeneous with respect to the total enthalpy and entropy (or it is reduced on a homogeneous jet using the technique described above). The flow in them is therefore potential. The boundary of the jets is a tangential discontinuity in which the x and y coordinates, the stream function ψ, the angle ϑ, the pressure p and $\Delta = p - p_e = p - 1$ are continuous. The tangential discontinuity is a streamline of both jets. In the Δ, ϑ plane, they are identical and, consequently, when expansions (2.6) are used in a small neighbourhood of the origin of the coordinates, the solutions for ψ of the two jets must be identical in the line $\vartheta/\Delta^n = \chi_b$. Variables at the tangential discontinuity will henceforth be labelled with the subscript b. Therefore, if the zeroth terms of expansions (2.6) for the two jets, that is, ψ_{01} and ψ_{02}, are non-zero (the meaning of this stipulation will become clear later), they must be functions of the one and the same χ and $n_1 = n_2 = n$.

So, the conditions

$$\psi_1(p, 0) = 0, \quad \psi_{1b} = m, \quad \psi_{2b} = m, \quad \psi_2(p_e, \vartheta) = 1, \quad \left(\frac{dx}{dp}\right)_{b2} = \left(\frac{dx}{dp}\right)_{b1} \tag{3.1}$$

with a specified constant m, $0 \leqslant m \leqslant 1$, are satisfied on the boundaries of the jets. The last equality in (3.1) is the condition for the continuity of the coordinate x at the tangential discontinuity. When it is satisfied, the continuity of the second coordinate follows from the continuity of ϑ and the equality (the tangential discontinuity is a streamline)

$$dy/dx = \operatorname{tg}\theta = -\operatorname{tg}\vartheta$$

We now consider different situations.

Both jets are subcritical. In this case, $p_{*1}, p_{*2} < p_e = 1$ and, of all the solutions (2.8) and (2.13) giving bounded ψ_1 and ψ_2, only the solution (2.8) with $n = 1$ is valid. In the case of this solution, $(d\vartheta/dp)_b = \chi_b$ at the tangential discontinuity. Hence, from relations (2.5) and (2.8), we obtain

$$\psi_{0k} = C_{1k}\operatorname{arctg}\frac{\chi}{\beta_{ek}} + C_{2k}, \quad \left(\frac{dx}{dp}\right)_{bk} = -\frac{C_{1k}\beta_{ek}V_{ek}}{\Delta_b}, \quad \chi = \frac{\vartheta}{\Delta} \tag{3.2}$$

Taking account of this, from the last condition of (3.1) we find

$$C_{12}\beta_{e2}V_{e2} = C_{11}\beta_{e1}V_{e1}$$

On substituting here the constants C_{11} and C_{12}, found from the first four conditions of (3.1), we arrive at the equality

$$m\beta_{e1}V_{e1}\left(\operatorname{arctg}\frac{\chi_b}{\beta_{e2}} - \frac{\pi}{2}\right) + (1-m)\beta_{e2}V_{e2}\operatorname{arctg}\frac{\chi_b}{\beta_{e1}} = 0 \tag{3.3}$$

for determining χ_b as well as the functions m ($0 \leqslant m \leqslant 1$) and the other parameters of the jets. The left-hand side of this equality is a bounded, continuous and monotonically increasing function of χ_b. As χ_b varies from zero (in the plane of symmetry) to plus infinity (on the boundary with the submerged space), it increases from a known negative value up to a positive value, which is also known. Equation (3.3) therefore has a unique root $0 \leqslant \chi_b \leqslant \infty$ with limiting values corresponding either to a single external jet ($m = 0$) or a single internal jet ($m = 1$).

The internal jet is subcritical and the external jet is critical. In this case, from solutions (2.8) for a subcritical "internal" (lower) jet ($k = 1, p_{*1} < 1, 0 \leqslant \chi \leqslant \chi_b < \infty$) and solutions (2.13) for a critical "external" (upper) jet ($k = 2, p_{*2} = 1, 0 < \chi_b \leqslant \chi \leqslant \infty$), bounded ψ_{01} and ψ_{02} are also only obtained when $n = 1$ and $\chi = \vartheta/\Delta$. Taking the solution from (2.13) with $n = 1 < 3/2$, we calculate $(dx/dp)_{b2}$ for it using (2.11). Together with solution (3.2) for the subcritical jet ($k = 1$), this, when account is taken of the conditions on the axis and in the external boundary of the jet, gives

$$\psi_{01} = C_{11}\operatorname{arctg}\frac{\chi}{\beta_{e1}}, \quad \left(\frac{dx}{dp}\right)_{b1} = -\frac{C_{11}\beta_{e1}V_{e1}}{\Delta_b}, \quad \psi_{02} = \frac{C_{12}}{\chi} + 1, \quad \left(\frac{dx}{dp}\right)_{b2} = \frac{C_{12}}{\Delta_b}, \quad \chi = \frac{\vartheta}{\Delta}$$

Hence, as in the preceding case, from the conditions at the tangential discontinuity, we arrive at the following equation for the determining χ_b

$$\frac{m}{\chi_b} - \frac{1-m}{\beta_{e1}V_{e1}}\operatorname{arctg}\frac{\chi_b}{\beta_{e1}} = 0$$

The proof of the fact that, when $0 < m < 1$, it has a unique solution $0 < \chi_b < \infty$ is even simpler than in the case of Eq. (3.3).

In both of the cases considered above, the solutions for a jet close to the axis are identical to the solution presented in Section 2. Hence, the first formula of (2.15), which expresses x_e in terms of $\ln(\Delta/\Delta_0)$, holds, apart from a constant factor, and the equalizing with respect to p and ϑ is asymptotic.

The internal jet is critical and the external jet is subcritical. This case is of particular interest as solution with $\psi_{01} \neq 0$ and $\psi_{02} \neq 0$ as well as with $\psi_{02} \equiv 0$ when $\psi_{01} \neq 0$ cannot successfully be constructed for it. If it is assumed that $\psi_{01} \equiv 0$, then a solution in the neighbourhood of the singular point is constructed but not for any m and $V_{e2} < 1$. Hence, this example demonstrates the limited nature of the approach being used.

In the case of a critical internal jet ($k = 1, p_{*1} = 1, 0 \leqslant \chi \leqslant \chi_b \leqslant \infty$), the bounded solution from (2.13) for ψ_{01} corresponds to $n_1 \geqslant 3/2$ and, in the case of the subcritical external jet ($k = 2, p_{*2} < 1, 0 < \chi_b \leqslant \chi \leqslant \infty$), the bounded solution from (2.8) for ψ_{02} corresponds to $n_2 \leqslant 1$. The impossibility of choosing the same exponent in the variable χ for both jets enables one (according to the stipulation made at the start of Section 3) to postulate that ψ_{01} or ψ_{02} is identically zero. We shall suppose that $\psi_{01} \equiv 0$ and take account of the fact that, in the Δ, ϑ plane, the tangential discontinuity and the disrupted jet separate the internal jet from the ϑ-axis. This fact enables one, when representing the stream function of the internal jet in the form (2.6) with $\psi_{01} \equiv 0$, to take $\delta(\Delta, \vartheta) = \Delta^l$ with a positive exponent $l > 0$, which is unknown in advance, as the distance to the singular point. In accordance with this

$$\psi_1 = \psi_{11}(\chi)\Delta^l + \ldots, \quad \chi = \vartheta\Delta^{-n} \tag{3.4}$$

Substituting expression (3.4) into Eq. (2.10) and retaining the leading terms of the expansion, we arrive at the equation

$$l(l-1)\psi_{11} + n(1+n-2l)\chi\psi'_{11} + n^2\chi^2\psi''_{11} + (3q/2)^2\Delta^{3-2n}\psi''_{11} = 0 \tag{3.5}$$

When $n \neq 3/2$, its solutions have the form

$$\psi_{11}(\chi) = C_{11}\chi + C_{21}, \quad n > \frac{3}{2}; \quad \psi_{11}(\chi) = C_{11}\chi^{(l-1)/n} + C_{21}\chi^{l/n}, \quad n < \frac{3}{2} \tag{3.6}$$

The equation, which is obtained from (3.5) when $n = \dfrac{3}{2}$,

$$9(\chi^2 + q^2)\psi''_{11} + 3(5-4l)\chi\psi'_{11} + 4l(l-1)\psi_{11} = 0 \tag{3.7}$$

has solutions, when l differs from 0 and 1, which are expressed in terms of Legendre functions with pure imaginary last variable (i/q). The case when $l = 0$ is interesting since, for such l, expansion (3.4) does not differ from (2.6) and, when $l = 1$, the solution of Eq. (3.7) is

$$\psi_{11}(\chi) = C_{11}I(\chi_0, \chi, q) + C_{21}, \quad n = \frac{3}{2}, \quad l = 1, \quad \chi = \vartheta/\Delta^{3/2}$$

Of the corresponding n different solutions of Eq. (3.5) for a flow with a critical internal jet and a subcritical external jet, only (3.6) is valid and, moreover, for any $l > 0$. In fact, by virtue of relations (3.4) and (3.6) and the condition in the plane of symmetry $\psi_1(0) = 0$, for the internal we have $C_{21} = 0$ and

$$\psi_1(\chi) = \Delta^l\psi_{11}(\chi) = \Delta^l C_{11}\chi = C_{11}\vartheta/\Delta^{n-l}, \quad n > \frac{3}{2}, \quad l > 0 \tag{3.8}$$

In order to match this solution for the critical internal jet with the last solution of (2.8) with $\chi = \vartheta/\Delta$ for the subcritical external jet, it is necessary to put $n - l = 1$ in equality (3.8); the conditions which are imposed on n and l allow this. So, after some obvious changes in notation, we obtain

$$\psi_1(\chi) = C_{11}\chi, \quad \psi_2(\chi) = C_{12}\operatorname{arctg}\frac{\chi}{\beta_{e2}} + C_{22}, \quad \chi = \frac{\vartheta}{\Delta} \tag{3.9}$$

From this and from equalities (2.5) and (2.11), we find

$$\left(\frac{dx}{dp}\right)_{b1} = -\frac{C_{11}\chi_b^2}{\Delta_b}, \quad \left(\frac{dx}{dp}\right)_{b2} = -\frac{C_{12}\beta_{e2}V_{e2}}{\Delta_b} \tag{3.10}$$

From relations (3.9) and (3.10) and conditions, (3.1), for determining χ_b, we obtain the equation

$$(1-m)\beta_{e2}V_{e2} + m\chi_b\left(\operatorname{arctg}\frac{\chi_b}{\beta_{e2}} - \frac{\pi}{2}\right) = 0 \tag{3.11}$$

When $m > V_{e2}/(1 + V_{e2})$, it has a unique, bounded, positive root which determines the asymptotic character of the equalizing of these jets. If $m = V_{e2}/(1 + V_{e2})$, the root of Eq. (3.1), $\chi_b = \infty$, is of no interest. When $m < V_{e2}/(1 + V_{e2})$, Eq. (3.11) does not have positive roots.

If the first solution of (2.8) ($n < 1$) is taken for the external jet, then, within the framework of a similar approach, the solution $\psi_1(\chi)$ of (3.4) cannot be successfully joined to it for any m and $V_{e2} < 1$. For $\psi_{02} \equiv 0$, when $\psi_{01} \neq 0$, solutions satisfying the last condition of (3.1) do not exist for any values of m and $V_{e2} < 1$.

Two critical jets. According to what has been said at the start of Section 2, the analysis of piecewise-homogeneous gas jets with the same physical properties and different critical pressures reduces to the analysis of a homogeneous jet. We shall therefore consider jets of two different gases; two perfect gases with adiabatic exponents κ_1 and κ_2, for example, In this case, $p_{*1}, = p_{*2} = 1$ and, from all of the solutions

(2.8) and (2.13) which give bounded ψ_1 and ψ_2, only the last solution of (2.13) ($n = 3/2$) is suitable for which $(d\vartheta/dp)_b = 3\chi_b\Delta_b^{1/2}/2$ on the tangential discontinuity. From this and from relations (2.11) and (2.13), when account is taken of a part of conditions (3.1), we obtain

$$\psi_{01} = C_{11}I(0, \chi, q_1), \quad \psi_{02} = C_{12}I(\chi_b, \chi, q_2) + m, \quad \left(\frac{dx}{dp}\right)_{bk} = -\frac{9C_{1k}}{4\Delta_b^{1/2}}(\chi_b^2 + q_k^2)^{1/6}, \quad \chi = \frac{\vartheta}{\Delta^{3/2}}$$

Taking account of these last relations, on writing out condition (3.1) with the integration constants C_{11} and C_{12}, which are determined by the conditions of (3.1) which have not been used up to now, we arrive at the equation

$$\varphi(\chi_b) \equiv mI(\chi_b, \infty, q_2) - (1-m)I(0, \chi_b, q_1)\xi_1(\chi_b) = 0, \quad \xi_j(\chi_b) = \left(\frac{\chi_b^2 + q_{j+1}^2}{\chi_b^2 + q_j^2}\right)^{1/6} \tag{3.12}$$

The left-hand side of Eq. (3.12) is a continuous function of χ_b. When χ_b varies from 0 to ∞, it decreases from the positive value $mI(0, \infty, q_2)$ to the negative value $(m-1)I(0, \infty, q_1)$. Consequently, this equation has just a single root χ_b, $0 < \chi_b < \infty$, and χ_b increases from 0 to ∞ as m increases from 0 to 1. If $q_1 \geqslant q_2$, then the monotonicity of the decrease is demonstrated practically at once and there is one root. When $q_1 < q_2$, the proof of the uniqueness of the root is made difficult by the decrease of the factor $\xi_1(\chi_b)$. However, this factor, if it also increases, is insignificant. For instance, in the case of perfect gases with the limiting values of the adiabatic exponents: $\kappa_1 = 5/3$ and $\kappa_2 = 1$, it increases from 1 to 1.096 as χ_b changes from 0 to ∞, Its small growth does not modify the monotonic decrease in $\varphi(\chi_b)$ and the conclusion concerning the uniqueness of the root of Eq. (3.12) still holds. Irrespective of the dependence on the number of roots, formulae (2.16), which determine the finite abscissa of the cross-section of the equalizing of the jets with respect to pressure, the angle of inclination of the velocity and the Mach number still hold (with other factors accompanying $\Delta_0^{1/2}$ or $\vartheta_0^{1/3}$).

Three critical jets. The case of three critical gas jets which differ in their physical properties (in the case of perfect gases, this means that they have different adiabatic exponents) can be treated in a similar manner. As in the preceding case, the flow in each jet is described by the last equation of (2.13) ($n = 3/2$). Suppose m_1 and m_2 are the specified values of the stream function at the first (closest to the plane of symmetry) and at the second tangential discontinuity ($0 \leqslant m_1 \leqslant m_2 \leqslant 1$), and $\chi_{b1} \leqslant \chi_{b2}$ are the values of the variable χ corresponding to them. Then, to determine χ_{b1} and χ_{b2} in a similar way to Eq. (3.12), we obtain the two equations

$$\begin{aligned} \varphi_1(\chi_{b1}, \chi_{b2}) &\equiv m_1 I(\chi_{b1}, \chi_{b2}, q_2) - (m_2 - m_1)I(0, \chi_{b1}, q_1)\xi_1(\chi_b) = 0 \\ \varphi_2(\chi_{b1}, \chi_{b2}) &\equiv (m_2 - m_1)I(\chi_{b2}, \infty, q_3) - (1 - m_2)I(\chi_{b1}, \chi_{b2}, q_2)\xi_2(\chi_b) = 0 \end{aligned} \tag{3.13}$$

We also write out their corollary

$$\varphi_3(\chi_{b1}, \chi_{b2}) \equiv m_1 I(\chi_{b2}, \infty, q_3) - (1 - m_2)I(0, \chi_{b1}, q_1)\xi_1(\chi_b)\xi_2(\chi_b) = 0 \tag{3.14}$$

which, as would be expected, reduces to Eq. (3.12) on "removal" of the middle jet. This is achieved if one puts

$$m_1 = m_2 = m, \quad \chi_{b1} = \chi_{b2} = \chi_b$$

in Eq. (3.14) and then replaces q_3 with q_2. On such removal of the middle jet ($m_1 = m_2$, $\chi_{b1} = \chi_{b2}$) and the second equation of (3.13) is identically satisfied. Equation (3.12) can also be obtained from the first equation of (3.13) if we put

$$m_1 = m, \quad m_2 = 1, \quad \chi_{b1} = \chi_b, \quad \chi_{b2} = \infty$$

in (3.13).

The transition from three to two jets can also be accomplished using the corresponding substitutions in the second equation of (3.13).

Suppose m_1 and m_2 are specified and that $0 < m_1 < m_2 < 1$. Then, when χ_{b1} varies from 0 up to any $\chi_{b2} > 0$, the sign of $\varphi_1(\chi_{b1}, \chi_{b2})$, that is, the left-hand side of Eq. (3.13), changes, where χ_{b1} and χ_{b2} are continuous functions. If, however, $\chi_{b2} = 0$, then, simultaneously, $\chi_{b1} = 0$, and, if $\chi_{b2} \to \infty$, then

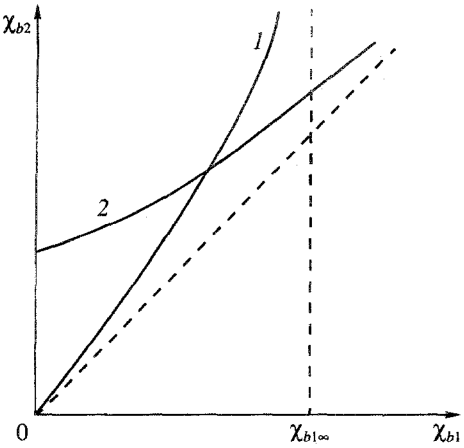

Fig. 2

$\chi_{b1} \to \chi_{b1\infty} < \infty$. Hence, in the χ_{b1}, χ_{b2} plane, in which the upper half of the first quadrant ($0 \leqslant \chi_{b1} \leqslant \chi_{b2} \leqslant \infty$), emerging from the origin of the coordinates and located above the line $\chi_{b1} = \chi_{b2}$ corresponds to the domain of permissible values, the line of the solutions of the first equation of (3.13): $\varphi_1(\chi_{b1}, \chi_{b2}) = 0$, for the specified values of m_1 and m_2 departs to infinity with respect to χ_{b2} and has a vertical asymptote (curve *1* in Fig. 2)

When χ_{b2} varies from any $\chi_{b1} \geqslant 0$ to ∞, the sign of $\varphi_2(\chi_{b1}, \chi_{b2})$, that is, the left-hand side of the second equation of (3.13) changes, $0 < \chi_{b2} < \infty$ when $\chi_{b1} = 0$ and $\chi_{b2} \to \infty$ when $\chi_{b1} \to \infty$. Starting out on the ordinate, the line of solutions of the second equation of (3.13): $\varphi_2(\chi_{b1}, \chi_{b2}) = 0$ for the specified m_1 and m_2, which is also located above the line $\chi_{b1} = \chi_{b2}$, therefore departs to infinity both with respect to χ_{b1} and χ_{b2} (curve 2 in Fig. 2). The point of intersection of curves *1* and *2* then gives the solution of both of equations (3.13).

Its numerical solution also confirmed the validity of the conclusion which has been drawn concerning the roots of system (3.13). Here, the inverse problem was solved. We found q_1^2, q_2^2 and q_3^2 in accordance with the last formulae of (2.10) for the different adiabatic exponents κ_1, κ_2 and κ_3. Then, using formulae (3.6), the surfaces $m_1 = m_1(\chi_{b1}, \chi_{b2})$ and $m_2 = m_2(\chi_{b1}, \chi_{b2})$ were constructed for χ_{b1} and χ_{b2} belonging to the upper half of the first quadrant of the χ_{b1}, χ_{b2} plane and which consequently satisfy the inequalities

$$0 \leq \chi_{b1} \leq \chi_{b2} \leq \infty$$

In accordance with what has been said earlier, the surfaces which have been constructed in all of the examples for which calculations have been carried out satisfied the conditions $0 < m_1 \leqslant m_2 < 1$, with an equality sign only when $\chi_{b1} = \chi_{b2}$.

Hence, the equalizing of a three-layer critical jet with respect to p and ϑ, as in the case of a two-layer jet, occurs at a finite distance from the nozzle exit.

4. CONCLUSION

The above analysis can be transferred practically without any change to other jet problems. In this sense, the problem considered earlier [5, 8–10] of the flow of a homogeneous, potential, sonic jet into a symmetric, wedge-like obstacle and its piecewise-potential generalization is the simplest. Problems of asymmetric efflux with an angle of inclination of the equalized jet θ_e which is unknown in advance and undetermined within the framework of the approach used, are somewhat more complex. However, a knowledge of θ_e is not required to analyse the limiting properties. It suffices to put $\vartheta = \theta_e - \theta$ and, also, to take account of the fact that, when there is no plane of symmetry, the integration constants are found from the conditions on the two external boundaries of the jet. At the same time, the example in Section 3 with an internal critical jet and an external subcritical jet is indicative of the possible limited nature of this approach.

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